Some Non-trivial Symmetry Classes of Tensors Associated with Certain Characters

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Abstract

Let G be a finite group and Ω a set of n elements. Assume that G acts faithfully on Ω and let V be a vector space over the complex field $\mathbb{C}$, with $\dim V = m \geq 2$. It is shown that for each irreducible constituent χ of permutation character of G, the symmetry class of tensors associated with G and χ is non-trivial. This extends a result of Merris and Rashid (see [6, Theorem 2]).

Keywords: Symmetry class of tensors, Permutation character, Irreducible constituent.

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1 Introduction

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$, and let G be a subgroup of the symmetric group on n letters, $\$_n$, and let χ be an irreducible complex character of G. Suppose $\phi : \overset{n}{\times} V \to U$ be an n-linear function, where U is a finite dimensional vector space over $\mathbb{C}$. We say ϕ is *symmetric* with respect to G and χ if for all $v_1, \ldots, v_n \in V$,

$$\frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \phi(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}) = \phi(v_1, \ldots, v_n).$$

A finite dimensional vector space S over $\mathbb{C}$ is called a *symmetry class of tensors* associated with G and χ if there is an n-linear function $\phi : \overset{n}{\times} V \to S$ which is symmetric with respect to G and χ such that

(i) $\langle \mathrm{Im}\ \phi \rangle = S$,

(ii) for each finite dimensional vector space U over $\mathbb{C}$ and for each n-linear function $\psi : \overset{n}{\times} V \to U$, symmetric with respect to G and χ, there exists a unique linear transformation $f : S \to U$ such that the following diagram commutes.

$$\begin{tikzcd}
\overset{n}{\times} V \arrow[r, "\phi"] \arrow[d, "\psi"'] & S \arrow[dl, "f"] \\
U &
\end{tikzcd}$$

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One easily obtains that the symmetry class of tensors associated with G and χ exists and it is unique up to isomorphisms of vector spaces. We denote the symmetry class of tensors associated with G and χ by $V_\chi^n(G)$. Let $\overset{n}{\otimes}V$ be the n-th tensor power of V and write $v_1 \otimes \ldots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in $\$_n$ there corresponds a unique linear operator $P(\sigma) : \overset{n}{\otimes}V \longrightarrow \overset{n}{\otimes}V$ determined by

$$P(\sigma)(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

Now if we define the linear operator $T(G,\chi) : \overset{n}{\otimes}V \longrightarrow \overset{n}{\otimes}V$ by

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma),$$

then one easily obtains that the symmetry class of tensors associated with G and χ is isomorphic with the image of $T(G,\chi)$ and so we can assume that the symmetry class of tensors associated with G and χ is equal to this image: $V_\chi^n(G) = T(G,\chi)(\overset{n}{\otimes}V)$. The image of $v_1 \otimes \ldots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \ldots * v_n$ and it is called a *decomposable symmetrized tensor*. Let $\mathrm{Irr}(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\Big\{ T(G,\chi) : \overset{n}{\otimes}V \longrightarrow \overset{n}{\otimes}V \Big| \, \chi \in \mathrm{Irr}(G) \Big\},$$

is a set of annihilating idempotents which sum to the identity and therefore by a classical theorem in linear algebra we have

$$\bigotimes^n V = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi^n(G).$$

It is well-known that

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) m^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ (see [5]).

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leq \alpha_i \leq m$, so α is a mapping from a set of n elements into a set of m elements. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$ where $\sigma \in G$ is a permutation on n letters and $\alpha \in \Gamma_m^n$ is a mapping from a set of n elements into a set of m elements. Therefore the action may be written as $\sigma \cdot \alpha = \alpha\sigma^{-1}$ which is a composition of two functions.

Let $O(\alpha) = \{\sigma \cdot \alpha |\ \sigma \in G\}$ be the *orbit* with representative α, also let G_α be the *stabilizer subgroup* of α, i.e., $G_\alpha = \{\sigma \in G |\ \sigma \cdot \alpha = \alpha\}$, and consider Δ be a system of distinct representatives of the orbits of Γ_m^n, when G acts on Γ_m^n.

Consider $\{e_1, \ldots, e_m\}$ be a basis of V. Denote by e_α^* the tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$. For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma\cdot\alpha}^* |\ \sigma \in G\rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha\in\Delta} V_\alpha^*. \tag{1}$$

Note that, it is possible for some $\alpha \in \Delta$ we have $V_\alpha^* = 0$. But Freese (see [3]) proved that for $\alpha \in \Delta$

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma\in G_\alpha} \chi(\sigma), \tag{2}$$

and therefore, if we consider

$$\overline{\Delta} = \Big\{\alpha \in \Delta |\ \sum_{\sigma\in G_\alpha} \chi(\sigma) \neq 0\Big\},$$

then the sum appeared in (1) will be reduced to the form of

$$V_\chi^n(G) = \bigoplus_{\alpha\in\overline{\Delta}} V_\alpha^*, \tag{3}$$

of course we define that the right hand of (3) equals 0, if $\overline{\Delta} = \emptyset$.

Non-triviality of $V_\chi^n(G)$ were studied by several authors. In [1], the authors discuss the non-triviality of the symmetry class of tensors associated with the projective special linear group, $PSL_2(q)$, as a subgroup of the symmetric group $\$_{q+1}$. Also in [2], the authors considered G as a subgroup of $\$_n$ by Cayley representation and proved that if $m \geq 2$, then for all χ, $\chi \in \mathrm{Irr}(G)$, $V_\chi^n(G) \neq 0$. The main purpose of this article is to generalize a result of Merris and Rashid (see [6, Theorem 2]). In fact, they proved that if G is a doubly transitive subgroup of $\$_n$ and $\chi = \theta - 1_G$, where θ is permutation character of G, then for all m, $m \geq 2$, $V_\chi^n(G) \neq 0$.

In the next section, we will prove that the condition of "doubly transitivity of G" can be omitted and so, when $m \geq 2$, the symmetry class of tensors associated with G and each constituent of its permutation character is non-trivial. This generalization will be independent of the permutation structure of the group G and we will obtain the result appeared in [2] that we mentioned earlier.

2 Results

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$ and let G be a finite group and Ω a set of n elements. Suppose G acts faithfully on Ω, so we can assume that G is a subgroup of $\$_n$. In fact, we consider $\{f_\sigma |\ \sigma \in G\}$ as the group G where $f_\sigma : \Omega \longrightarrow \Omega$, defined by $f_\sigma(\omega) = \sigma \cdot \omega$ for all $\omega \in \Omega$, is a permutation on n letters. Therefore the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in \mathrm{Irr}(G)$. Denote the permutation character of G by θ. For $\sigma \in G$, the value $\theta(\sigma)$ is the number of letters fixed by σ, i.e., $\theta(\sigma) = |\{\omega \in \Omega |\ \sigma \cdot \omega = \omega\}|$.

Main Theorem *Let G be a finite group and Ω a set of n elements. Assume that G acts faithfully on Ω and let V be an m-dimensional vector space over the complex field $\mathbb{C}$. If $m \geq 2$, then for all irreducible constituent χ of permutation character of G, we have $V_\chi^n(G) \neq 0$.*

Proof. If $n = 1$, the theorem is obvious, so we assume that $n \geq 2$. Denote the permutation character of G by θ and suppose χ be any irreducible constituent of θ, therefore $(\theta, \chi)_G \neq 0$. Without losing generality, we can assume that $\Omega = \{1, 2, \ldots, n\}$. Consider $\{i_1, \ldots, i_s\} \subseteq \Omega$ be a system of distinct representatives of the orbits of Ω, when G acts on Ω. It is well-know that

$$\theta = 1_{G_{i_1}} \uparrow^G + \ldots + 1_{G_{i_s}} \uparrow^G,$$

where G_{i_j}, $1 \leq j \leq s$, is the stabilizer subgroup of i_j (see [4, Lemma 5.14]). Therefore $(\theta, \chi)_G \neq 0$ implies that

$$(1_{G_{i_1}} \uparrow^G, \chi)_G + \ldots + (1_{G_{i_s}} \uparrow^G, \chi)_G \neq 0,$$

and so for some j, $1 \leq j \leq s$,

$$(1_{G_{i_j}} \uparrow^G, \chi)_G \neq 0. \tag{4}$$

Now we consider $\alpha = (1, \ldots, 1, 2, 1, \ldots, 1)$ where 2 appeared in i_j-th place. Since $n \geq 2$ and $m \geq 2$, we obtain that $\alpha \in \Gamma_m^n$. Consider the action of G on Γ_m^n and choose Δ such that $\alpha \in \Delta$. Note that $G_\alpha = G_{i_j}$, where G_α is the stabilizer subgroup of α when G acts on Γ_m^n and G_{i_j} is the stabilizer subgroup of i_j when G acts on Ω. So by Frobenius reciprocity we have

$$(1_{G_{i_j}} \uparrow^G, \chi)_G = (1_{G_\alpha} \uparrow^G, \chi)_G = (\chi \downarrow_{G_\alpha}, 1_{G_\alpha})_{G_\alpha} = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma),$$

and by (4) we obtain that

$$\sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0.$$

Therefore $\alpha \in \overline{\Delta}$ and so we have $\overline{\Delta} \neq \emptyset$. Now by (2) and (3) we obtain that $V_\chi^n(G)$ $\neq 0$. $\square$

Corollary 1 ([Theorem 2 of 6]) *Let G be a doubly transitive subgroup of $\$_n$. Let $\chi = \theta - 1_G$, where θ is permutation character of G. If dim$V = m \geq 2$, then $V_\chi^n(G) \neq 0$.*

Proof. G has a natural faithful action on set $\Omega = \{1, \ldots, n\}$ by $\sigma \cdot i = \sigma(i)$. Doubly transitivity of this action implies that $\chi \in \mathrm{Irr}(G)$. So χ is an irreducible constituent of θ and therefore by the main theorem we have $V_\chi^n(G) \neq 0$. $\square$

Now we consider a special case. Suppose G is a group of order n and G acts on G by right multiplication, i.e., for all $g, g' \in G$, $g.g' = gg'$. This action is faithful, therefore G is a subgroup of $\$_n$ and in this case we say G is a subgroup of $\$_n$ by *Cayley representation.*

Corollary 2 ([Theorem 5 of 2]) *Let G be a group of order n, that is a subgroup of $\$_n$ by Cayley representation. If dim$V = m \geq 2$, then for all χ, $\chi \in \mathrm{Irr}(G), V_\chi^n(G) \neq 0$.*

Proof. When G is a subgroup of $\$_n$ by Cayley representation, each elements of G are fixed point free, except identity, therefore permutation character of G, in this case, is regular character and so any irreducible character of G is a constituent for θ. So by main theorem for all χ, $\chi \in \mathrm{Irr}(G), V_\chi^n(G) \neq 0$. $\square$

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